Lecture 13

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1 Dimension and basis of the span

Last lecture we formulated the problem of finding the basis and the dimension of the span of given vectors. This lecture we will give the algorithm to determine these characteristics of the span.

1.1 Algorithm

Step 1. Write the given vectors as a rows of a matrix.

Step 2. Reduce this matrix to REF keeping track which row in the matrix corresponds to which vector, i.e. initially, i-th row corresponds to u_i – so, label i-th row as i, and after Type 1 elementary operation we interchange row labels.

Step 3. The number of nonzero rows is a dimension of a span. The labels of nonzero rows are subscripts of vectors in basis. Moreover, nonzero rows form a basis for a span as well.

Example 1.1. *Consider the following 4 vectors in $\mathbb{R}^4$: $u_1 = (2, 3, 1, 1)$, $u_2 = (1, 1, 0, -1)$, $u_3 = (3, 4, 1, 0)$, and $u_4 = (1, 2, 1, 3)$. Let's find the basis and a dimension of their span. To do* *this we'll form a matrix with labels and reduce it to REF.*

$$\begin{array}{c} 1\\2\\3\\4 \end{array}\begin{pmatrix} 2 & 3 & 1 & 1\\ 1 & 1 & 0 & -1\\ 3 & 4 & 1 & 0\\ 1 & 2 & 1 & 3 \end{pmatrix} \underset{\leadsto}{\textit{interchange 1st and 2nd}} \begin{array}{c} 2\\1\\3\\4 \end{array}\begin{pmatrix} 1 & 1 & 0 & -1\\ 2 & 3 & 1 & 1\\ 3 & 4 & 1 & 0\\ 1 & 2 & 1 & 3 \end{pmatrix} \underset{\leadsto}{\textit{subtract 1st from others}}$$

$$\begin{array}{c} 2\\1\\3\\4 \end{array}\begin{pmatrix} 1 & 1 & 0 & -1\\ 0 & 1 & 1 & 3\\ 0 & 1 & 1 & 3\\ 0 & 1 & 1 & 4 \end{pmatrix} \underset{\leadsto}{\textit{subtract 2nd from others}} \begin{array}{c} 2\\1\\3\\4 \end{array}\begin{pmatrix} 1 & 1 & 0 & -1\\ 0 & 1 & 1 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{\leadsto}{\textit{interchange 3rd and 4th}}$$

$$\begin{array}{c} 2\\1\\4\\3 \end{array}\begin{pmatrix} 1 & 1 & 0 & -1\\ 0 & 1 & 1 & 3\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, the number of nonzero rows is 3, so, the dimension of a span is 3. Moreover, we see that vectors u_1, u_2 *and* u_4 *form a basis of the span, and moreover vectors* $(1,1,0,-1)$, $(0,1,1,3)$ *and* $(0,0,0,1)$ *form another basis of a span.*

Using this last basis formed by nonzero rows of a matrix in REF we can determine whether a given vector belongs to the span or not.

Example 1.2. *Consider a vector* $(1,2,3,4)$ *and let's find whether this vector belongs to the span from the previous example. We have a basis with 3 vectors:* $(1,1,0,-1)$, $(0,1,1,3)$ *and* $(0,0,0,1)$. *Let's try to express our given vector as a linear combination of these 3 basis vectors:*

$$(1,2,3,4) = a(1,1,0,-1) + b(0,1,1,3) + c(0,0,0,1) = (a, a+b, b, -a+3b+c).$$

So, $a = 1$, *then* $a + b = 2$, *so* $b = 2$. *Now when we're considering the third component, we see that* $b = 3$. *So, we got a contradiction. So, vector* $(1,2,3,4)$ *cannot be expressed as a linear combination of 3 vectors from the basis, and so it is not in basis.*

Now consider vector $(5,7,2,0)$. *Let's try to express our given vector as a linear combination of 3 basis vectors:*

$$(5,7,2,0) = a(1,1,0,-1) + b(0,1,1,3) + c(0,0,0,1) = (a, a+b, b, -a+3b+c).$$

So, $a = 5$, *then* $a + b = 7$, *so* $b = 2$. *Now when we're considering the third component, we see that* $b = 2$ *– that's ok, no contradiction, and then* $-a+3b+c = 0 \Leftrightarrow -5+6+c = 0 \Leftrightarrow c = -1$. *So,*

$$(5,7,2,0) = 5(1,1,0,-1) + 2(0,1,1,3) - (0,0,0,1),$$

and thus this vector belongs to the span of initial vectors.

1.2 Explanation of the algorithm

When we considered elementary operations on vectors, we proved that they do not change the span. From this fact it follows that span of nonzero rows of the REF is the same as span of initial vectors. Moreover, the nonzero rows of REF are linearly independent — if we form a linear combination of them which is equal to $\mathbf{0}$, we'll get that all the coefficients are equal to 0 (it's very easy check!). So, nonzero rows of REF form a basis of a span, and since a dimension is a number of vectors in basis, it is equal to the number of nonzero rows.

The proof that the labels of nonzero rows are subscripts of the vectors in basis is a little bit more tricky. We'll give the idea. By performing elementary operations, we add vectors, multiplied by different numbers to the other vectors. If the row with label i became a zero row, it means that we added some vectors multiplied by numbers to the i-th vector, and got 0. So, we can express i-th vector as a linear combination of other vectors, and we should exclude it from the basis. So, finally we'll get that vectors with subscripts which are equal to the labels of nonzero rows of REF are linearly independent. Moreover, the number of them is equal to the dimension. So, they form a basis.

2 Rank of a matrix

Definition 2.1. *The **rank** of a matrix is a number of nonzero rows in its REF (or RREF).*

Example 2.2. *Let* $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. *To transpose it to REF we multiply the first row by 2 and subtract it from the second one. So, REF is equal to* $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. *The number of nonzero rows is equal to* 1. *So,* $\operatorname{rk}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 1$.

Example 2.3. *Let* $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$. *To transpose it to REF we multiply the first row by 2 and subtract it from the second one. So, REF is equal to* $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. *The number of nonzero rows is equal to* 2. *So,* $\operatorname{rk}\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = 2$.

Let's think what does this value stand for. Last lecture we solved a problem of computing the dimension of a span of vectors. To compute the dimension of a span we had the following algorithm. We wrote the vectors as rows of a matrix, transposed it to a REF, and the number of nonzero rows was equal to the dimension of a span of these vectors. So, we see that for any given matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

its rank

$$\operatorname{rk} A = \dim \operatorname{span}(a_1, a_2, \dots, a_m),$$

where

$$\begin{aligned} a_1 &= (a_{11}, a_{12}, \dots, a_{1n}); \\ a_2 &= (a_{21}, a_{22}, \dots, a_{2n}); \\ &\dots; \\ a_m &= (a_{m1}, a_{m2}, \dots, a_{mn}). \end{aligned}$$